UNIQUENESS OF GROUND STATES OF SOME COUPLED NONLINEAR SCHRODINGER SYSTEMS AND THEIR APPLICATION

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Abstract. We establish the uniqueness of ground states of some coupled nonlinear Schrodinger systems in the whole space. We firstly use Schwartz symmetrization to obtain the existence of ground states for a more general case. To prove the uniqueness of ground states, we use the radial symmetry of the ground states to transform the systems into an ordinary differential system, and then we use the integral forms of the system. More interestingly, as an application of our uniqueness results, we derive a sharp vector-valued Gagliardo-Nirenberg inequality.

Keywords: Schrodinger system, uniqueness of ground states, sharp vector-valued Gagliardo-Nirenberg inequality

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1. Introduction

In this paper we are concerned with the uniqueness of ground states of the coupled nonlinear Schrödinger system:

$$-i\partial_t\phi_j = \Delta\phi_j + \mu_j|\phi_j|^{2p}\phi_j + \sum_{i\neq j}\beta_{ij}|\phi_i|^{p+1}|\phi_j|^{p-1}\phi_j, \tag{1}$$

where $\phi_j = \phi_j(t,x) \in \mathbb{C}$, $x \in \mathbb{R}^n$, $t > 0$, $j = 1, ..., N$. Here $0 < p < 2/(n-2)^+$ (we use the convention: $2/(n-2)^+ = +\infty$ when $n = 1, 2$, and $(n-2)^+ = n-2$ when $n \geq 3$), μ_j's and β_{ij}'s are coupling constants subjected to $\beta_{ij} = \beta_{ji}$.

The model (1) has applications in many physical problems, especially in nonlinear optics. An application of (1) comes from [1], the solution ϕ_j denotes the j^{th} component of the beam in Kerr-like photo-refractive media. The constant μ_j is for self-focusing in the j^{th} component of the beam. The coupling constant β_{ij} is the interaction between the i^{th} and the j^{th} component of the beam. We refer to [4] for more precision on the meaning of the constants. Another application of (1) arises in [10]. When two optical waves of different frequencies co-propagate in a medium and interact nonlinearly through the medium, or when two polarization components of a wave interact nonlinearly at some central frequency, the propagation equations for

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the two problems can be considered together as the following N coupled nonlinear Schrödinger-like equations for the case $N=2$:

$$i\partial_t\phi_j+\partial_x^2\phi_j+\kappa_j\phi_j+\left(\sum_{i=1}^N p_{ij}|\phi_i|^2\right)\phi_j+\left(\sum_{i=1}^N q_{ij}\phi_i^2\right)\bar{\phi_j}=0,$$

where $j=1,...,N$, ϕ_j denotes the complex amplitude of the j^{th} electric field envelope, or the j^{th} polarization component, p_{ij}'s , q_{ij}'s and κ_j's are parameters characteristic of the medium and interaction. Especially, when $\kappa_j=0$ and $q_{ij}=0$, it reduces to our model problem (1) where $n=1$ and $p=1$.

To obtain solitary solutions of the system (1), we set $\phi_j(t,x)=e^{it}u_j(x)$ ($u_j\in\mathbb{R}$) and transform the system (1) to steady-state N coupled nonlinear Schrödinger equations given by

$$u_j-\Delta u_j=\mu_j|u_j|^{2p}u_j+\sum_{i\neq j}\beta_{ij}|u_i|^{p+1}|u_j|^{p-1}u_j,\quad j=1,...,N. \tag{2}$$

The concept of incoherent solitary solutions have attracted considerable attentions in the last ten years, both from experimental and theoretical point of view. The two experimental studies [20] and [22] demonstrated the existence of solitary waves made from both spatially and temporally incoherent light. These papers were followed by a large amount of theoretical work on incoherent solitary waves, see for example [4, 10, 11] and the references therein. The energy functional of (2) is

$$\begin{aligned}\mathcal{E}(\mathbf{u}):=&\frac{1}{2}\sum_{i=1}^N\int_{\mathbb{R}^n}\left(|\nabla u_i|^2+u_i^2\right)-\frac{1}{2p+2}\sum_{i=1}^N\int_{\mathbb{R}^n}\mu_i u_i^{2p+2}\\&-\frac{1}{2p+2}\sum_{i,j=1}^N\int_{\mathbb{R}^n}\beta_{ij}|u_i|^{p+1}|u_j|^{p+1}.\end{aligned}$$

This functional is well defined if $u_i\in H^1(\mathbb{R}^n)$, by virtue of the embedding $H^1(\mathbb{R}^n)\hookrightarrow L^{2p+2}(\mathbb{R}^n)$ with $0<p<2/(n-2)^+$. We will always consider solitary waves with finite energy, and will be particularly interested in the least energy nontrivial solutions of (2), which are named ground states in Physics.

Let's recall some previous work about the ground states of (2) related to our research in this paper. In order to simplify the presentation, we shall concentrate on the system of two equations:

$$\begin{cases} u_1-\Delta u_1=\mu_1|u_1|^{2p}u_1+\beta|u_2|^{p+1}|u_1|^{p-1}u_1,\\ u_2-\Delta u_2=\mu_2|u_2|^{2p}u_2+\beta|u_1|^{p+1}|u_2|^{p-1}u_2.\end{cases} \tag{3}$$

A solution $\mathbf{u}=(u_1,u_2)$ of (3) is called nontrivial if $u_1\not\equiv 0$ and $u_2\not\equiv 0$ simultaneously. The nontrivial weak solutions of (3) are equivalent to the

nontrivial critical points of the energy functional

$$\begin{aligned}\mathcal{E}(\mathbf{u}) =&\frac{1}{2}\int_{\mathbb{R}^n}\left(|\nabla u_1|^2+u_1^2+|\nabla u_2|^2+u_2^2\right)\\&-\frac{1}{2p+2}\int_{\mathbb{R}^n}\left(\mu_1 u_1^{2p+2}+2\beta|u_1|^{p+1}|u_2|^{p+1}+\mu_2 u_2^{2p+2}\right)\end{aligned}$$

in the Sobolev space $H:=H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)$. Notice that any nontrivial solution of (3) has to belong to the Nehari manifold

$$\begin{aligned}\mathcal{N}:=\{\mathbf{u}\in H, u_1\not\equiv 0, u_2\not\equiv 0;\quad &\int_{\mathbb{R}^n}\left(|\nabla u_1|^2+u_1^2+|\nabla u_2|^2+u_2^2\right)\\=\int_{\mathbb{R}^n}&\left(\mu_1 u_1^{2p+2}+2\beta|u_1|^{p+1}|u_2|^{p+1}+\mu_2 u_2^{2p+2}\right)\}.\end{aligned}$$

Definition 1. *The nonnegative minima of the minimization problem*

$$c:=\inf_{\mathbf{u}\in\mathcal{N}}\mathcal{E}(\mathbf{u}) \tag{4}$$

is called the ground state of (3).

In the case of a single nonlinear Schrödinger equation, the ground state exists [3] and was proved to be the positive solution of

$$\Delta u-u+u^{2p+1}=0. \tag{5}$$

The positive solution of (5) is radial symmetric about some fixed point [8] and is unique in the sense of moduling translations [12]. We denote it by ω hereafter.

Quite differently from the case of a single equation, the existence of ground states solutions with multi-components of the system (2) is much more complicated than the single case and was studied quite well when $\mu_j>0$ in the series of the papers [2, 15, 21, 23]. Roughly speaking, they proved that there always exist ranges of positive parameters u_j's, β_{ij}'s in (2), for which this system has a least energy solution, and ranges of positive parameters for which the energy functional can't be minimized on the Nehari manifold where the eventual solutions lie. Readers can consult these papers for further details.

However, the uniqueness of positive solutions of the system (2) is a widely open problem, and to our knowledge no results have been already known in this direction. In our present paper, we discuss the ground state of (2) in the case $\mu_j\le 0$, which has not been considered before, and we will prove that in this case the ground state is unique. The uniqueness of ground states when $\mu_j>0$ remains open. To be precise, our result reads as follows.

Theorem 2. *Consider the steady-state two coupled nonlinear Schrödinger equations in* $\mathbb{R}^n$

$$\begin{cases} u_1-\Delta u_1=\mu_1|u_1|^{2p}u_1+\beta_1|u_2|^{p+1}|u_1|^{p-1}u_1,\\ u_2-\Delta u_2=\mu_2|u_2|^{2p}u_2+\beta_2|u_1|^{p+1}|u_2|^{p-1}u_2,\end{cases} \tag{6}$$

in which $0 < p < 2/(n-2)^+$. *Assume that*

$$\mu_1, \mu_2 \le 0, \quad \beta_1, \beta_2 > 0, \quad \mu_1 \beta_1^p = \mu_2 \beta_2^p, \tag{7}$$

and

$$\mu_1 + \frac{\beta_2^{(p+1)/2}}{\beta_1^{(p-1)/2}} > 0 \quad \text{or} \quad \mu_2 + \frac{\beta_1^{(p+1)/2}}{\beta_2^{(p-1)/2}} > 0. \tag{8}$$

Then the ground state of (6) exists and is unique up to translations. Moreover, the ground state can be determined explicitly by

$$\begin{cases} u_1 = \left(\mu_1 + \frac{\beta_2^{(p+1)/2}}{\beta_1^{(p-1)/2}}\right)^{-1/2p} \omega, \\ u_2 = \left(\mu_2 + \frac{\beta_1^{(p+1)/2}}{\beta_2^{(p-1)/2}}\right)^{-1/2p} \omega, \end{cases} \tag{9}$$

where ω *is the unique positive solution of (5).*

As far as we know, there are only two results about the uniqueness of positive solutions to stationary Schrodinger systems. One is in [13]. The other one is in [16], where the radial symmetry and uniqueness results have been obtained for the non-negative solutions to the schrodinger system

$$(I - \Delta)u = v^p, (I - \Delta)v = u^q.$$

One may see [6] for related uniqueness result.

We now give some remarks about the conditions (7) and (8).

Remark 3. *The condition (7) implies that*

$$\left(\mu_2 + \frac{\beta_1^{(p+1)/2}}{\beta_2^{(p-1)/2}}\right) = \frac{\beta_1^p}{\beta_2^p} \left(\mu_1 + \frac{\beta_2^{(p+1)/2}}{\beta_1^{(p-1)/2}}\right),$$

and hence in (8)

$$\mu_1 + \frac{\beta_2^{(p+1)/2}}{\beta_1^{(p-1)/2}} > 0 \Leftrightarrow \mu_2 + \frac{\beta_1^{(p+1)/2}}{\beta_2^{(p-1)/2}} > 0.$$

Remark 4. *It's easy to check that the special case*

$$\mu_1 = \mu_2 = \mu \le 0, \quad \beta_1 = \beta_2 = \beta \quad \text{and} \quad \mu + \beta > 0 \tag{10}$$

satisfies (7) and (8). Conversely, any other constants satisfying (7) and (8) can be transformed to the case (10) by scaling. In fact, set

$$w_1(x) := a_1 u_1(x), \quad w_2(x) := a_2 u_2(x),$$

where $a_1, a_2 > 0$ *are the scaling constants. Then the equations (6) can be written as*

$$\begin{cases} w_1 - \Delta w_1 = \frac{\mu_1}{a_1^{2p}} w_1^{2p+1} + \frac{\beta_1}{a_1^{p-1} a_2^{p+1}} w_1^p w_2^{p+1}, \\ w_2 - \Delta w_2 = \frac{\mu_2}{a_2^{2p}} w_2^{2p+1} + \frac{\beta_2}{a_2^{p-1} a_1^{p+1}} w_2^p w_1^{p+1}. \end{cases}$$

The condition (7) guarantees the existence of $a_1, a_2 > 0$ *such that*

$$\frac{\mu_1}{a_1^{2p}} = \frac{\mu_2}{a_2^{2p}} := \mu, \quad \frac{\beta_1}{a_1^{p-1}a_2^{p+1}} = \frac{\beta_2}{a_2^{p-1}a_1^{p+1}} := \beta.$$

Indeed, we can choose without loss of generality that

$$a_1 = 1, \quad a_2 = (\frac{\beta_1}{\beta_2})^{1/2}$$

and thus

$$\mu = \mu_1, \quad \beta = \frac{\beta_2^{(p+1)/2}}{\beta_1^{(p-1)/2}}.$$

The condition (8) is just that $\mu + \beta > 0$.

Remark 5. *The relation* $\mu + \beta > 0$ *in (10) plays a crucial role to ensure that the Nehari manifold* $\mathcal{N} \neq \emptyset$.

As is well known, the sharp Gagliardo-Nirenberg inequality plays extremely important roles in the quantitative analysis of blow-up solutions of the single Schrödinger equation. A large amount of work relies heavily on the sharp constant in the Gagliardo-Nirenberg inequality. We shall only quote here [18, 19, 24, 25, 26] where a comprehensive list of references on this subject can be found. As an application of Theorem 2, we can obtain a sharp vector-valued Gagliardo-Nirenberg inequality. To our experience, the sharp vector-valued Gagliardo-Nirenberg inequality we obtain here would play some non-negligible roles in further studies of Schrödinger systems.

Corollary 6. *Let* $0 < p < 2/(n-2)^+$ *and* $\mathcal{K}_{n,p}$ *be the sharp constant in the single valued Gagliardo-Nirenberg inequality, that is,*

$$\|u\|_{2p+2}^{2p+2} \leq \mathcal{K}_{n,p}\|u\|_2^{2p+2-np}\|\nabla u\|_2^{np}, \quad \forall\ u \in H^1(\mathbb{R}^n).$$

Assume the constants μ, β *satisfy*

$$\mu \leq 0, \quad \text{and} \quad \mu + \beta > 0.$$

Then we have the two vector-valued Gagliardo-Nirenberg inequality below. That is, $\forall\ u_1, u_2 \in H^1(\mathbb{R}^n)$,

$$\begin{aligned} \mu\|u_1\|_{2p+2}^{2p+2} &+ 2\beta\|u_1u_2\|_{p+1}^{p+1} + \mu\|u_2\|_{2p+2}^{2p+2} \\ &\leq \mathcal{K}_{n,p,\mu,\beta}\left(\|u_1\|_2^2 + \|u_2\|_2^2\right)^{p+1-np/2}\left(\|\nabla u_1\|_2^2 + \|\nabla u_2\|_2^2\right)^{np/2}, \end{aligned} \tag{11}$$

in which the sharp constant $\mathcal{K}_{n,p,\mu,\beta}$ *is determined by*

$$\mathcal{K}_{n,p,\mu,\beta} = \frac{(\mu+\beta)}{2^p}\mathcal{K}_{n,p}.$$

Remark 7. *If one uses Hölder inequality directly, one can only get a vector-valued Gagliardo-Nirenberg inequality like*

$$\begin{aligned}
&\mu\|u_1\|_{2p+2}^{2p+2} + 2\beta\|u_1u_2\|_{p+1}^{p+1} + \mu\|u_2\|_{2p+2}^{2p+2} \\
&\leq (\mu+\beta)\left(\|u_1\|_{2p+2}^{2p+2} + \|u_2\|_{2p+2}^{2p+2}\right) \\
&\leq (\mu+\beta)\left(\|u_1\|_{2p+2}^{2} + \|u_2\|_{2p+2}^{2}\right)^{p+1} \\
&\leq (\mu+\beta)\mathcal{K}_{n,p}\left(\sum_{j=1}^{2}(\|u_j\|_2^2)^{\frac{p+1-np/2}{p+1}}(\|\nabla u_j\|_2^2)^{\frac{np/2}{p+1}}\right)^{p+1} \\
&\leq (\mu+\beta)\mathcal{K}_{n,p}\left((\|u_1\|_2^2 + \|u_2\|_2^2)^{\frac{p+1-np/2}{p+1}}(\|\nabla u_1\|_2^2 + \|\nabla u_2\|_2^2)^{\frac{np/2}{p+1}}\right)^{p+1} \\
&= (\mu+\beta)\mathcal{K}_{n,p}\left(\|u_1\|_2^2 + \|u_2\|_2^2\right)^{p+1-np/2}\left(\|\nabla u_1\|_2^2 + \|\nabla u_2\|_2^2\right)^{np/2},
\end{aligned}$$

in which the constant $(\mu+\beta)\mathcal{K}_{n,p}$ is in strong contrast with the sharp constant $(\mu+\beta)\mathcal{K}_{n,p}/2^p$. In fact, the sharp constant relies heavily on the explicit expressions of ground states.

Remark 8. *Somewhat surprisingly, our arguments to prove Theorem 2 and corollary 6 can't be generalized to the N coupled Schrödinger system with $N \geq 3$. So we have to leave the case $N \geq 3$ as an open problem. For the scalar Gagliardo-Nirenberg inequality in its general form, one may see E.Hebey's book* [9].

To prove Theorem 2, we only deal with the standard case (10) of (6), as what we have explained in Remark 4. In section 2, we use Schwartz symmetrization to prove that the minimization problem (4) can be achieved by a positive solution of the system (6), which indicates the existence of ground states in a more general case. In section 3, we transform (6) to a system of ordinary differential equations (ODE) by virtue of the radial symmetry of positive solutions of (6). Then by the comparison technique of ODE, we arrive at the uniqueness of positive solutions of (6). Since all the ground sates must be positive solutions of (6), we conclude that the ground state is unique. In section 4, we prove the sharp vector-valued Gagliardo-Nirenberg inequality (Corollary 6) in detail.

2. Existence of ground states

This section is devoted to the proof of the existence of ground states of (6) in the case

$$\mu_1, \mu_2 \leq 0 \ \text{ and } \ \mu_1 + \beta > 0, \quad \mu_2 + \beta > 0. \tag{12}$$

The existence of ground states of (6) when $\mu_1, \mu_2 > 0$ has been extensively studied in the papers [2, 15, 21, 23] using the method of Schwartz symmetrization. We declare that this symmetrization method still works for

the case (12) under our consideration. Since our proof would have many details different from the ones in the preceding papers, we will give our proof thoroughly for the purpose of completeness. We point out that our proof, which combines the analysis in [2] and [15], could be seen as a simplified version of their arguments.

We have the following proposition, which asserts that all the critical points of the minimization problem (4) must be weak solutions of (6) in H.

Proposition 9. *If the minimization problem (4) is attained by a coupled* $\mathbf{u} \in \mathcal{N}$, *then* $\mathbf{u}$ *is a solution of (6).*

Proof. The proof of Proposition 9 is similar to the one in [2]. Let

$$\begin{aligned}\mathcal{G}(\mathbf{u}) := &\int_{\mathbb{R}^n} \left(|\nabla u_1|^2 + u_1^2 + |\nabla u_2|^2 + u_2^2\right) \\ &- \int_{\mathbb{R}^n} \left(\mu_1 u_1^{2p+2} + 2\beta |u_1|^{p+1}|u_2|^{p+1} + \mu_2 u_2^{2p+2}\right).\end{aligned}$$

We have for each $\psi = (\psi_1, \psi_2) \in H$ that

$$\begin{aligned}< \nabla\mathcal{E}(\mathbf{u}), \psi >= &\sum_{i=1}^{2} \int_{\mathbb{R}^n} \left(\nabla u_i \cdot \nabla \psi_i + u_i \psi_i - \mu_i u_i^{2p+1} \psi_i\right) \\ &- \sum_{i=1}^{2} \int_{\mathbb{R}^n} \beta |u_i|^{p-1} |u_j|^{p+1} u_i \psi_i, \quad j \neq i,\end{aligned}$$

$$\begin{aligned}< \nabla\mathcal{G}(\mathbf{u}), \psi >= &2\sum_{i=1}^{2} \int_{\mathbb{R}^n} \left(\nabla u_i \cdot \nabla \psi_i + u_i \psi_i - (p+1)\mu_i u_i^{2p+1} \psi_i\right) \\ &- 2(p+1)\sum_{i=1}^{2} \int_{\mathbb{R}^n} \beta |u_i|^{p-1} |u_j|^{p+1} u_i \psi_i. \quad j \neq i.\end{aligned}$$

Suppose that $\mathbf{u} = (u_1, u_2) \in \mathcal{N}$ is a minimizer for $\mathcal{E}$ restricted on $\mathcal{N}$, then the standard minimization theory yields an Euler-Lagrange multiplier $L \in \mathbb{R}$ such that

$$\nabla\mathcal{E}(\mathbf{u}) + L\nabla\mathcal{G}(\mathbf{u}) = 0.$$

Setting $\mathcal{G}(\mathbf{u}) =< \nabla\mathcal{E}(\mathbf{u}), \mathbf{u} >= 0$ in the expression $< \nabla\mathcal{E}(\mathbf{u}) + L\nabla\mathcal{G}(\mathbf{u}), \mathbf{u} >= 0$, we obtain that

$$L \int_{\mathbb{R}^n} \left(|\nabla u_1|^2 + u_1^2 + |\nabla u_2|^2 + u_2^2\right) = 0,$$

which implies that $L = 0$, thanks to $\mathbf{u} \not\equiv 0$. $\square$

Next, we use Schwartz symmetrization to prove that the minimum c in (4) can be achieved by a positive solution of (6) as in [15]. The following lemma [14] is at the heart of our argument.

Lemma 10. *Let u^* be the Schwartz symmetric function associated to u, namely the radially symmetric, radially non-increasing function, equi-measurable with u. There hold for $1 \le p < \infty$ that*

$$\int_{\mathbb{R}^n} |\nabla u^*|^2 \le \int_{\mathbb{R}^n} |\nabla u|^2, \quad \forall\, u \in H^1(\mathbb{R}^n), \quad u \ge 0;$$
$$\int_{\mathbb{R}^n} |u^*|^p = \int_{\mathbb{R}^n} |u|^p, \quad \forall\, u \in L^p(R^n), \quad u \ge 0;$$
$$\int_{\mathbb{R}^n} (u^*)^p (v^*)^p \ge \int_{\mathbb{R}^n} u^p v^p, \quad \forall\, u, v \in L^{2p}(\mathbb{R}^n), \quad u, v \ge 0.$$

After these preparations, we now state and prove the main result in this section.

Theorem 11. *Assume (12). Then the ground states of (6) exist and are positive solutions of (6).*

Proof. With the help of Proposition 9, we see it remains only to verify that the minimum of c in (4) can be attained by a pair of positive functions in $\mathcal{N}$. Define

$$\overline{\mathcal{N}} := \{\mathbf{u} \in H, u_1 \not\equiv 0, u_2 \not\equiv 0; \quad \int_{\mathbb{R}^n} \left(|\nabla u_1|^2 + u_1^2 + |\nabla u_2|^2 + u_2^2\right)$$
$$\le \int_{\mathbb{R}^n} \left(\mu_1 u_1^{2p+2} + 2\beta |u_1|^{p+1} |u_2|^{p+1} + \mu_2 u_2^{2p+2}\right)\},$$

and

$$\overline{c} := \inf_{\mathbf{u} \in \overline{\mathcal{N}}} \mathcal{E}(\mathbf{u}).$$

The conditions $\mu_1 + \beta > 0$, $\mu_2 + \beta > 0$ ensure that $\mathcal{N} \neq \emptyset$, $\overline{\mathcal{N}} \neq \emptyset$. It's obviously that $\overline{c} \le c$.

Step 1. Noting that

$$\mathcal{E}(\mathbf{u}) \ge \frac{p}{2p+2} \int_{\mathbb{R}^n} \left(|\nabla u_1|^2 + u_1^2 + |\nabla u_2|^2 + u_2^2\right) > 0, \quad \forall\, \mathbf{u} \in \overline{\mathcal{N}},$$

the definition of $\overline{c}$ makes sense. Since $\mathcal{E}(u_1, u_2) = \mathcal{E}(|u_1|, |u_2|)$, we can take a nonnegative minimizing sequence $\{\mathbf{u}_k\}$ of $\overline{c}$. We use C to denote various constants independent of $\mathbf{u_k}$. By Sobolev embedding and $\mu_1, \mu_2 \le 0$, it follows that for all $\mathbf{u}_k \in \overline{\mathcal{N}}$ that

$$\begin{aligned}
&\|u_{k,1}\|_{2p+2} \|u_{k,2}\|_{2p+2} \\
\le\ & \frac{1}{2}(\|u_{k,1}\|_{2p+2}^2 + \|u_{k,2}\|_{2p+2}^2) \\
\le\ & C \int_{\mathbb{R}^n} \left(|\nabla u_{k,1}|^2 + u_{k,1}^2 + |\nabla u_{k,2}|^2 + u_{k,2}^2\right) \\
\le\ & C \int_{\mathbb{R}^n} \left(\mu_1 u_{k,1}^{2p+2} + 2\beta |u_{k,1}|^{p+1} |u_{k,2}|^{p+1} + \mu_2 u_{k,2}^{2p+2}\right) \\
\le\ & C\beta \|u_{k,1}\|_{2p+2}^{p+1} \|u_{k,2}\|_{2p+2}^{p+1},
\end{aligned}$$

which implies that $\|u_{k,1}\|_{2p+2}\|u_{k,2}\|^{2p+2} \geq C > 0$. Let $\mathbf{u}_k^* = (u_{k,1}^*, u_{k,2}^*)$ be the Schwartz symmetrization of $\mathbf{u}_k$. By lemma 10 one checks easily that

$$\mathbf{u}_k^* \in \overline{\mathcal{N}}, \quad \|u_{k,1}^*\|_{2p+2}\|u_{k,2}^*\|_{2p+2} \geq C > 0, \tag{13}$$

and $\mathbf{u}_k^*$ is also a minimizing sequence of $\overline{c}$. By the well-known compact embedding from radial symmetric functions in $H^1(\mathbb{R}^n)$ to $L^{2p+2}(\mathbb{R}^n)$ [25], one can assume that $\mathbf{u}_k^* \to \mathbf{u}^*$ in $L^{2p+2}(\mathbb{R}^n) \times L^{2p+2}(\mathbb{R}^n)$. By Fatou's lemma, $\mathbf{u}^* \in \overline{\mathcal{N}}$, and

$$\overline{c} = \mathcal{E}(\mathbf{u}^*).$$

Moreover, from (13), we deduce that $u_1^* \not\equiv 0, u_2^* \not\equiv 0$.

Step 2. We claim that

$$\begin{aligned} &\int_{\mathbb{R}^n} \left(|\nabla u_1^*|^2 + u_1^{*2} + |\nabla u_2^*|^2 + u_2^{*2}\right) \\ = &\int_{\mathbb{R}^n} \left(\mu_1 u_1^{*2p+2} + 2\beta |u_1^*|^{p+1}|u_2^*|^{p+1} + \mu_2 u_2^{*2p+2}\right). \end{aligned}$$

Suppose not, we have

$$\begin{aligned} &\int_{\mathbb{R}^n} \left(|\nabla u_1^*|^2 + u_1^{*2} + |\nabla u_2^*|^2 + u_2^{*2}\right) \\ < &\int_{\mathbb{R}^n} \left(\mu_1 u_1^{*2p+2} + 2\beta |u_1^*|^{p+1}|u_2^*|^{p+1} + \mu_2 u_2^{*2p+2}\right). \end{aligned}$$

Then $\mathbf{u}^*$ belongs to the interior of $\overline{\mathcal{N}}$, that is, $\mathbf{u}^*$ is an interior critical point of $\mathcal{E}(\mathbf{u})$, and this leads to

$$\nabla \mathcal{E}(\mathbf{u}^*) = 0,$$

which implies that $\mathbf{u}^*$ is a weak solution of (6). Multiplying (6) by $\mathbf{u}^*$ and integrating over $\mathbb{R}^n$ by parts, we have

$$\begin{aligned} &\int_{\mathbb{R}^n} \left(|\nabla u_1^*|^2 + u_1^{*2} + |\nabla u_2^*|^2 + u_2^{*2}\right) \\ = &\int_{\mathbb{R}^n} \left(\mu_1 u_1^{*2p+2} + 2\beta |u_1^*|^{p+1}|u_2^*|^{p+1} + \mu_2 u_2^{*2p+2}\right), \end{aligned}$$

which is a contradiction.

Step 3. From Step 1 and Step 2, we have that

$$c = \overline{c} = \mathcal{E}(\mathbf{u}^*).$$

By Proposition 9, $\mathbf{u}^*$ is a nonnegative solution of (6) such that $u_1^* \not\equiv 0, u_2^* \not\equiv 0$. The maximum principle applied to each single equation in (6) suggests that $u_1^* > 0, u_2^* > 0$ and the proof of the existence of ground states of (6) is finished.

Step 4. We assert that all the ground states must be positive solutions of (6). In fact, Proposition 9 demonstrates that all the ground states are nonnegative solutions of (6) and each component of the solutions is nonzero. By the strong maximum principle, these solutions must be strictly positive. The proof of Theorem 11 is complete. □

3. Uniqueness of ground states

We are now in position to prove the uniqueness of the ground states in the case

$$\mu_1 = \mu_2 = \mu \le 0, \quad \text{and} \quad \mu + \beta > 0.$$

The positive weak solutions of (6) in H when $\beta > 0$ were proved to be regular enough, be radial symmetric up to translations, and decay to zero exponentially as $|x| \to +\infty$ in [5, 16]. If we denote

$$u_1(x) = u_1(|x|) = u_1(r), \quad u_2(x) = u_2(|x|) = u_2(r),$$

we are then led to the following ODE system:

$$\left\{ \begin{array}{l} -(r^{n-1}u_1')' + r^{n-1}u_1 = \mu r^{n-1}u_1^{2p+1} + \beta r^{n-1}u_1^p u_2^{p+1}, \\ -(r^{n-1}u_2')' + r^{n-1}u_2 = \mu r^{n-1}u_2^{2p+1} + \beta r^{n-1}u_2^p u_1^{p+1}. \end{array} \right. \tag{14}$$

By the radial symmetry again we have that

$$u_1'(0) = u_2'(0) = 0.$$

Integrating (14) from 0 to r we have

$$\left\{ \begin{array}{l} u_1'(r) = r^{1-n}\int_0^r t^{n-1}u_1 - \mu r^{1-n}\int_0^r t^{n-1}u_1^{2p+1} - \beta r^{1-n}\int_0^r t^{n-1}u_1^p u_2^{p+1}, \\ u_2'(r) = r^{1-n}\int_0^r t^{n-1}u_2 - \mu r^{1-n}\int_0^r t^{n-1}u_2^{2p+1} - \beta r^{1-n}\int_0^r t^{n-1}u_2^p u_1^{p+1}. \end{array} \right. \tag{15}$$

Integrating once again from 0 to r we achieve

$$\left\{ \begin{array}{ll} u_1(r) & = u_1(0) + \int_0^r t^{1-n}\int_0^t s^{n-1}u_1(s) - \mu\int_0^r t^{1-n}\int_0^t s^{n-1}u_1^{2p+1}(s) \\ & \quad -\beta\int_0^r t^{1-n}\int_0^t s^{n-1}u_1^p u_2^{p+1}(s), \\ u_2(r) & = u_2(0) + \int_0^r t^{1-n}\int_0^t s^{n-1}u_2(s) - \mu\int_0^r t^{1-n}\int_0^t s^{n-1}u_2^{2p+1}(s) \\ & \quad -\beta\int_0^r t^{1-n}\int_0^t s^{n-1}u_2^p u_1^{p+1}(s). \end{array} \right. \tag{16}$$

We claim that $u_1(0) = u_2(0)$. If else, suppose that $u_1(0) > u_2(0)$ for example, and define

$$R_0 := \sup_{R>0}\{R; \ \ \forall\, r \in (0, R), \ \ u_1(r) > u_2(r)\}.$$

We indicate that $R_0 = +\infty$. Otherwise, by continuity we have

$$u_1(R_0) = u_2(R_0). \tag{17}$$

However, from (16), and the facts that for all $s \in (0, R_0)$

$$\left\{ \begin{array}{l} u_1(s) - u_2(s) > 0, \\ -\mu\left(u_1^{2p+1}(s) - u_2^{2p+1}(s)\right) > 0, \\ -\beta\left(u_1^p u_2^{p+1}(s) - u_2^p u_1^{p+1}(s)\right) > 0, \end{array} \right.$$

we have

$$\begin{aligned}u_1(R_0)-u_2(R_0)=(u_1(0)-u_2(0))&+\int_0^{R_0}t^{1-n}\int_0^t s^{n-1}(u_1(s)-u_2(s))\\&-\mu\int_0^{R_0}t^{1-n}\int_0^t s^{n-1}(u_1^{2p+1}(s)-u_2^{2p+1}(s))\\&-\beta\int_0^{R_0}t^{1-n}\int_0^t s^{n-1}(u_1^pu_2^{p+1}(s)-u_2^pu_1^{p+1}(s))>0,\end{aligned}$$

which is a contradiction with (17). A further fact about u_1 and u_2 is that $(u_1-u_2)(r)$ is nondecreasing as r goes into infinity. Indeed, from (15) we have

$$\begin{aligned}u_1'(r)-u_2'(r)=r^{1-n}\int_0^r t^{n-1}(u_1-u_2)&-\mu r^{1-n}\int_0^r t^{n-1}(u_1^{2p+1}-u_2^{2p+1})\\&-\beta r^{1-n}\int_0^r t^{n-1}(u_1^pu_2^{p+1}-u_2^pu_1^{p+1})>0,\end{aligned}$$

where the inequality follows from $u_1>u_2$. Thus we have

$$\liminf_{r\to+\infty}(u_1-u_2)(r)\geq u_1(0)-u_2(0)>0,$$

which contradicts with the fact that $u_1,u_2\to 0$ as $r\to+\infty$. Similarly, one can show that $u_1(0)<u_2(0)$ is also impossible.

Now we have $u_1(0)=u_2(0)$ and $u_1'(0)=u_2'(0)=0$. We deduce from the standard uniqueness theory of the Cauchy problem of the ODE system that

$$u_1=u_2=u,$$

where u is the positive solution of

$$\Delta u-u+(\mu+\beta)u^{2p+1}=0.$$

Since the above equation has only one positive solution [12] up to translations given by

$$u=(\mu+\beta)^{-1/2p}\omega,$$

we arrive at the uniqueness of positive solutions of (6), and the proof of Theorem 2 is finished.

4. Sharp vector-valued Gagliardo-Nirenberg inequality

In this section, we derive the sharp vector-valued Gagliardo-Nirenberg inequality (Corollary 6) as an application of our uniqueness result of ground states and the method of [25] (see also [7]). We define the following manifold

$$\mathcal{M}:=\{u_1,u_2\in H^1(\mathbb{R}^n);\quad \mu\|u_1\|_{2p+2}^{2p+2}+2\beta\|u_1u_2\|_{p+1}^{p+1}+\mu\|u_2\|_{2p+2}^{2p+2}>0\},$$

and consider the minimization problem

$$\alpha:=\inf_{\mathbf{u}\in\mathcal{M}}\mathcal{J}(\mathbf{u}),$$

where

$$\mathcal{J}(\mathbf{u}) = \frac{(\|u_1\|_2^2 + \|u_2\|_2^2)^{p+1-np/2}(\|\nabla u_1\|_2^2 + \|\nabla u_2\|_2^2)^{np/2}}{\mu\|u_1\|_{2p+2}^{2p+2} + 2\beta\|u_1 u_2\|_{p+1}^{p+1} + \mu\|u_2\|_{2p+2}^{2p+2}}.$$

It's obvious that the sharp constant in the vector-valued Gagliardo-Nirenberg inequality (11) is

$$\mathcal{K}_{n,p,\mu,\beta} = \frac{1}{\alpha}.$$

Applying the same method exactly as in [25], we assert that the minimum of α can be achieved by a pair of positive solutions u_1^*, u_2^* of

$$\begin{cases} u_1^* - \Delta u_1^* = \mu {u_1^*}^{2p+1} + \beta {u_2^*}^{p+1} {u_1^*}^{p}, \\ u_2^* - \Delta u_2^* = \mu {u_2^*}^{2p+1} + \beta {u_1^*}^{p+1} {u_2^*}^{p}, \end{cases} \tag{18}$$

Multiplying (18) by $\mathbf{u}^*$ and integrating by parts over $\mathbb{R}^n$, we have

$$\|\nabla u_j^*\|_2^2 + \|u_j^*\|_2^2 = \mu\|u_j^*\|_{2p+2}^{2p+2} + \beta\|u_1^* u_2^*\|_{p+1}^{p+1},$$

which yields

$$\sum_{j=1}^{2} \|\nabla u_j^*\|_2^2 + \sum_{j=1}^{2} \|u_j^*\|_2^2 = \mu \sum_{j=1}^{2} \|u_j^*\|_{2p+2}^{2p+2} + 2\beta\|u_1^* u_2^*\|_{p+1}^{p+1}. \tag{19}$$

Moreover, the Pohozaev identity for (18) reads

$$\begin{aligned} &\frac{n-2}{2} \sum_{j=1}^{2} \|\nabla u_j^*\|_2^2 + \frac{n}{2} \sum_{j=1}^{2} \|u_j^*\|_2^2 \\ &= \frac{n}{2p+2} \left(\mu \sum_{j=1}^{2} \|u_j^*\|_{2p+2}^{2p+2} + 2\beta\|u_1^* u_2^*\|_{p+1}^{p+1} \right). \end{aligned} \tag{20}$$

From (19) and (20), we get that

$$\begin{cases} \left(\mu \sum_{j=1}^{2} \|u_j^*\|_{2p+2}^{2p+2} + 2\beta\|u_1^* u_2^*\|_{p+1}^{p+1}\right) = \frac{2p+2}{2p+2-np} \sum_{j=1}^{2} \|u_j^*\|_2^2, \\ \sum_{j=1}^{2} \|\nabla u_j^*\|_2^2 = \frac{np}{2p+2-np} \sum_{j=1}^{2} \|u_j^*\|_2^2, \end{cases}$$

which gives

$$\mathcal{J}(\mathbf{u}^*) = \frac{(np)^{np/2}(2p+2-np)^{1-np/2}}{2(p+1)} \left(\sum_{j=1}^{2} \|u_j^*\|_2^2 \right)^p.$$

Since we have already known by Theorem 2 that the positive solution of (18) is uniquely determined by

$$u_1^* = u_2^* = (\mu+\beta)^{-1/2p}\, \omega,$$

we arrive at

$$\mathcal{J}(\mathbf{u}^*) = \frac{(np)^{np/2}(2p+2-np)^{1-np/2}}{2(p+1)} \left(\frac{2}{(\mu+\beta)^{1/p}} \|\omega\|_2^2 \right)^p.$$

And therefore

$$\mathcal{K}_{n,p,\mu,\beta} = \frac{2(p+1)}{(np)^{np/2}(2p+2-np)^{1-np/2}\|\omega\|_2^{2p}} \cdot \frac{(\mu+\beta)}{2^p} = \frac{(\mu+\beta)}{2^p}\mathcal{K}_{n,p},$$

where the fact [25] that

$$\mathcal{K}_{n,p} = \frac{2(p+1)}{(np)^{np/2}(2p+2-np)^{1-np/2}\|\omega\|_2^{2p}}$$

is used, and this completes the proof of Corollary 6.

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